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Maximal entanglement of squeezed vacuum states via swapping with number-phase measurement

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We propose a method to refine entanglement via swapping from a pair of squeezed vacuum states by performing the Bell measurement of number sum and phase difference. The resultant states are maximally entangled by adjusting the two squeezing parameters to the same value. We then describe the teleportation of number states by using the entangled states prepared in this way.

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Recently, information technologies with quantum systems have widely been investigated. Among these advances, quantum teleportation was first proposed via a protocol by using a two-dimensional system [1]. This original protocol adopts internal degrees of freedom as Bell measurement variable, e.g., the polarization of light, while other protocols were proposed later to teleport the quadrature phase components [2,3] and the photon number states [3–5]. Furthermore, the concept of entanglement swapping, which transfers entanglement from one system to another, appeared in the verification experiment of teleportation [6].

In this paper, we address the quantum teleportation in the photon number states. Great efforts have been made so far, theoretically and experimentally, for the teleportation of photon polarization (two dimensional) and that of quadrature components (infinite dimensional). Then, it may be desired as an important future step to realize the teleportation in more than two (but finite) dimensions. The photon number states seem to be very promising for such a teleportation. The essential ingredients for teleportation are the EPR (Einstein-Podolsky-Rosen) resource and the Bell measurement. At present, squeezed vacuum states generated by parametric down-conversion are used as EPR resource, almost uniquely in the experiments utilizing photons, including the proposals of number state teleportation [3–5]. Strictly speaking, however, the squeezed vacuum state is not ideal, since it has nonuniform number distribution. Hence, the Fock space of number states should be restricted effectively to some finite-dimensional subspace in order to prepare physically, an ideal EPR resource for perfect teleportation. This is suitably achieved by the number sum measurement on two-photon modes. Furthermore, the phase difference is conjugate to the number sum, as shown later, and they naturally provide the joint Bell measurement.

In this way, it is realized that the number sum and phase difference measurements are essential for the multidimensional teleportation utilizing photons. We here formulate properly this Bell measurement by describing the simultaneous eigenstates of the Hermitian operators of number sum and phase difference. Then, we propose a different method to achieve the desired entanglement in the number basis. The Bell measurement plays a crucial role even for this purpose.

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Specifically, by performing the number-phase measurement on the relevant two-photon modes, the EPR states with more favorable distribution can be obtained via swapping from a pair of squeezed vacuum states. In particular, by adjusting the two squeezing parameters to the same value, we obtain the maximally entangled states via swapping (MESS) with uniform distribution. These MESS's can be used for a reliable teleportation protocol based on the number-phase Bell measurement.

A two-mode squeezed vacuum state is given as

$$|\lambda\rangle_{ab}=(1-\lambda^2)^{1/2}\sum_{n=0}^{\infty}\lambda^n|n\rangle_a|n\rangle_b, \tag{1}$$

where λ is the squeezing parameter. This nonuniform distribution with $\lambda<1$ in Eq. (1) is attributed to the fact that the Fock space is infinite dimensional, while the energy should be finite for the physical states. In contrast, we will show in the following that the physical EPR state of maximal entanglement with uniform distribution may be obtained if the Fock space is effectively cut off at certain photon number. We begin by taking the Bell basis (a complete set of orthonormal entangled states) for the number sum and phase difference in the two-mode Fock space $\{|n_a\rangle_a|n_b\rangle_b\}$ as

$$|N,m\rangle_{ab}=\sum_{k=0}^{N}\frac{[(\omega_{N+1}^{*})^m\alpha^*]^k}{\sqrt{N+1}}|N-k\rangle_a|k\rangle_b, \tag{2}$$

where $m=0,1,\ldots,N \bmod N+1$ ($|N,-1\rangle_{ab}\equiv|N,N\rangle_{ab}$, etc). The $(N+1)$ root to generate a Z_{N+1} is given by

$$\omega_{N+1}\equiv\exp[i2\pi/(N+1)], \quad (\omega_{N+1})^{N+1}=1. \tag{3}$$

A certain phase factor α ($|\alpha|=1$) is also introduced, which will be specified later. The transformation matrix $\mathcal{U}_{mk}^{(N)}\equiv[(\omega_{N+1}^{*})^m\alpha^*]^k/\sqrt{N+1}$ in Eq. (2) is really unitary due to the completeness of the $(N+1)$ roots $1,\omega_{N+1},\ldots,\omega_{N+1}^{N}$. These orthonormal Bell states $|N,m\rangle_{ab}$ are maximally entangled with $|\mathcal{U}_{mk}^{(N)}|=1/\sqrt{N+1}$, spanning the subspace $\{|N-k\rangle_a|k\rangle_b\}$ with number sum $N=n_a+n_b$. Hence, they serve promisingly as the basis for $(N+1)$-dimensional teleportation.

The essential point is that these Bell states are maximally entangled even in terms of the phase states defined by Pegg and Barnett [7],

$$|N,m\rangle_{ab}=\exp[-i\phi_0^{(N,a)}N]\sum_{m'=0}^{N}\frac{[(\omega_{N+1}^{*})^{m'+m}]^{N}}{\sqrt{N+1}}\times|\phi_{m'+m}^{(N,a)}\rangle_a|\phi_{m'}^{(N,b)}\rangle_b. \tag{4}$$

To derive this expression, we take the relations between the number states and the phase states $(p=a,b)$,

$$|n\rangle_p=\sum_{m=0}^{N}\frac{\exp[-in\phi_m^{(N,p)}]}{\sqrt{N+1}}|\phi_m^{(N,p)}\rangle_p, \tag{5}$$

$$|\phi_m^{(N,p)}\rangle_p=\sum_{n=0}^{N}\frac{\exp[in\phi_m^{(N,p)}]}{\sqrt{N+1}}|n\rangle_p. \tag{6}$$

The phases are defined with resolution $2\pi/(N+1)$ as

$$\phi_m^{(N,p)}=\phi_0^{(N,p)}+\frac{2\pi}{N+1}m. \tag{7}$$

Here, the superscript (N,p) denotes the fact that these phase eigenvalues may include the arbitrary reference phases $\phi_0^{(N,p)}$, possibly depending on the resolutions (N). By noting $\exp[i\phi_m^{(N,p)}]=(\omega_{N+1})^m\exp[i\phi_0^{(N,p)}]$ when substituting Eq. (5) into Eq. (2), the phase factor α is specified as

$$\alpha=\alpha_{ab}=\exp[i(\phi_0^{(N,a)}-\phi_0^{(N,b)})] \tag{8}$$

for reducing the phase factors coming from $\phi_0^{(N,a)}$ and $\phi_0^{(N,b)}$ to the overall $\exp[-i\phi_0^{(N,a)}N]$. Then, Eq. (4) is led by using the Z_{N+1} completeness with integer m,

$$\sum_{k=0}^{N}\frac{[(\omega_{N+1})^m]^k}{N+1}=\delta_{m0}^{(N+1)}=\begin{cases}1 & (m=0 \bmod N+1)\\ 0 & (m\neq 0 \bmod N+1).\end{cases} \tag{9}$$

The Bell measurement of number sum and phase difference is represented by the Hermitian operators,

$$\hat{N}_{a+b}\equiv\hat{N}_a+\hat{N}_b,\quad \hat{\phi}_{a-b}\equiv\sum_{N=0}^{\infty}[\hat{\phi}^{(N,a)}-\hat{\phi}^{(N,b)}]\hat{P}_N. \tag{10}$$

Here, $\hat{\phi}^{(N,a)}$ and $\hat{\phi}^{(N,b)}$ are the phase operators on the modes a and b, respectively, providing the eigenvalues of Eq. (7). The resolution $2\pi/(N+1)$ common to the modes a and b should be taken for the phase difference to be consistent with the number sum. The projection operator $\hat{P}_N$ extracts the states in the subspace $\{|N-k\rangle_a|k\rangle_b\}$ with number sum N. As seen clearly from Eqs. (2) and (4), the Bell states are the simultaneous eigenstates of number sum and phase difference,

$$\hat{N}_{a+b}|N,m\rangle=N|N,m\rangle, \tag{11}$$

$$\hat{\phi}_{a-b}|N,m\rangle=\phi_m^{(N,a-b)}|N,m\rangle, \tag{12}$$

where the phase difference eigenvalues are given by

$$\phi_m^{(N,a-b)}=[\phi_0^{(N,a)}-\phi_0^{(N,b)}]+\frac{2\pi}{N+1}m. \tag{13}$$

Since $[\hat{\phi}^{(N,a)}-\hat{\phi}^{(N,b)}]$ does not change the number sum N, it commutes with $\hat{P}_N$ as required for the Hermiticity of the entire phase difference operator $\hat{\phi}_{a-b}$. These results clarify that in the subspace with number sum N, the phase difference operator introduced by Luis and Sánchez-Soto [8] indeed coincides with the difference of the phase operators of the individual modes of Pegg and Barnett [7], while it is not separable in the entire two-mode Fock space. It is also obvious from Eqs. (11) and (12) that $\hat{N}_{a+b}$ and $\hat{\phi}_{a-b}$ are commutable:

$$[\hat{N}_{a+b},\hat{\phi}_{a-b}]=0. \tag{14}$$

Now we prepare a pair of squeezed vacuum states, 1-2 system and 3-4 system, for entanglement and teleportation. They can, in fact, be expressed in an entangled form via swapping (1-2, 3-4) $\rightarrow$ (1-4, 2-3) as

$$\begin{aligned}|\lambda\rangle_{12}|\lambda'\rangle_{34}&=(1-\lambda^2)^{1/2}(1-\lambda'^2)^{1/2}\\&\quad\times\sum_{N=0}^{\infty}\sum_{k=0}^{N}\lambda^{N-k}\lambda'^{k}|N-k\rangle_1|k\rangle_4|N-k\rangle_2|k\rangle_3\\&=(1-\lambda^2)^{1/2}(1-\lambda'^2)^{1/2}\\&\quad\times\sum_{N=0}^{\infty}\lambda^N\sum_{m=0}^{N}|N,m\rangle_{23}|N,-m\rangle_{14}^{(r)},\end{aligned} \tag{15}$$

with the generalized Bell states including the ratio of the squeezing parameters $r\equiv\lambda'/\lambda$,

$$|N,m\rangle_{14}^{(r)}=\sum_{k=0}^{N}r^k\frac{[(\omega_{N+1}^{*})^m\alpha_{14}^{*}]^k}{\sqrt{N+1}}|N-k\rangle_1|k\rangle_4, \tag{16}$$

where $\alpha_{14}=\alpha_{23}^{*}$. The relation (15) is derived by extending the sum over k with $\delta_{kk'}^{(N+1)}$ as

$$\sum_{k=0}^{N}(\cdots)(\alpha_{23}^{*}\alpha_{23})^k=\sum_{k=0}^{N}\sum_{k'=0}^{N}\delta_{kk'}^{(N+1)}(\cdots)(\alpha_{23}^{*})^k(\alpha_{23})^{k'}, \tag{17}$$

and by taking the completeness of Z_{N+1} for $\delta_{kk'}^{(N+1)}=\delta_{(k-k')0}^{(N+1)}$ as given in Eq. (9). Then the Bell states, maximally entangled with $\lambda=\lambda'$, are obtained in Eq. (15) by performing the number-phase Bell measurement on the 2-3 system, as shown in Fig. 1 as a part of quantum teleportation with MESS:

$$|\lambda\rangle_{12}|\lambda'\rangle_{34}\Rightarrow|N,-m\rangle_{14}^{(r)}\xrightarrow{\lambda=\lambda'}|N,-m\rangle_{14}. \tag{18}$$

The probability to obtain the result (N,m) with $\lambda=\lambda'$ in this Bell measurement is given from Eq. (15) by

$$P(N,m,\lambda)=(1-\lambda^2)^2\lambda^{2N}. \tag{19}$$

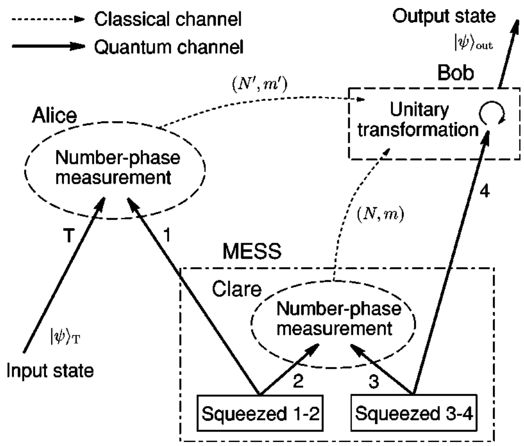

FIG. 1. Quantum teleportation with MESS. Alice and Bob share MESS, which is prepared by Clare with the number-phase measurement. Alice also performs the number-phase measurement. Bob receives classically the messages from Alice and Clare, and performs the unitary transformation to reconstruct the target state.

The original squeezed vacuum states have the nonuniform distribution with weight $\propto(\lambda^n)^2$. In contrast, the resultant Bell states $|N,-m\rangle_{14}$ with $\lambda=\lambda'$ are maximally entangled, having the uniform distribution in $(n_1,n_4)=(N-k,k)$ of the modes 1 and 4. Hence, they provide the ideal EPR resource for the teleportation of photon number states, as shown in Fig. 1. The target state may generally be prepared as

$$|\psi\rangle_{\text{T}}=\sum_{n=0}^{\infty} c_n|n\rangle_{\text{T}}. \tag{20}$$

Alice and Bob share the Bell state $|N,-m\rangle_{14}^{(r)}$ of the 1-4 system, which is prepared by Clare with the number-phase measurement on the modes 2 and 3 giving the result (N,m). Then, Alice performs the number-phase measurement on the modes T and 1 giving the result (N',m'), and the output state in the mode 4 is left for Bob as

$$\begin{aligned}|\psi'\rangle_{\text{out}}&={}_{1\text{T}}\langle N',m'||\psi\rangle_{\text{T}}|N,-m\rangle_{14}^{(r)}\\&=\sum_{n=n_0}^{N'}\frac{\exp[-i\phi(N,m,N',m')(n-\Delta N)]}{\sqrt{N+1}\sqrt{N'+1}}\\&\quad\times[(\omega_{N'+1})^{m'}\alpha_{1\text{T}}]^{\Delta N}r^{n-\Delta N}c_n|n-\Delta N\rangle_4,\end{aligned} \tag{21}$$

with $\Delta N\equiv N'-N$, $n_0\equiv\max[0,\Delta N]$ and

$$\phi(N,m,N',m')\equiv\arg[(\omega_{N'+1}^{*})^{m'}\alpha_{1\text{T}}^{*}(\omega_{N+1}^{*})^{m}\alpha_{14}]. \tag{22}$$

Here, due to the condition $0\leqslant n-\Delta N\leqslant N$ for the mode 4, the sum is taken over the photon number n as

$$n=\begin{cases}0,\ldots,N' & (N'\leqslant N)\\ \Delta N,\ldots,N' & (N'>N).\end{cases} \tag{23}$$

We have explicitly introduced Clare, who prepares the EPR resource, since in the present method, the entangled state is obtained depending on the result of Bell measurement. In contrast, it is usually the implicit recognition that one certain EPR resource, e.g., the singlet state of photon polarizations is prepared automatically by some device, which is known preceding to a teleportation experiment. In any case, the information on the EPR resource should be told to Bob.

Clare informs Bob which Bell state is prepared with the result (N,m) and Alice tells Bob the result (N',m') of the Bell measurement. Then, according to these classical messages, Bob performs a unitary transformation $|\psi\rangle_{\text{out}}=U_{\Delta N}U_{\phi}|\psi'\rangle_{\text{out}}$ to reconstruct the original state as faithfully as possible. The phase $\phi(N,m,N',m')$ can be removed by an operation with number operator as

$$U_\phi=\exp[i\hat{N}_4\phi(N,m,N',m')]. \tag{24}$$

Then, the number shift of ΔN can be made by another operation with phase operator [7] as

$$U_{\Delta N}=\exp[-i\hat{\phi}^{(\max[N,N'],4)}\Delta N]. \tag{25}$$

As a result, the output state is obtained as

$$|\psi\rangle_{\text{out}}=\frac{[(\omega_{N'+1})^{m'}\alpha_{1\text{T}}/r]^{\Delta N}}{\sqrt{N+1}\sqrt{N'+1}}\sum_{n=n_0}^{N'}r^n c_n|n\rangle_4. \tag{26}$$

That is, the number state teleportation is made as

$$c_n(0\leqslant n<\infty)\ \Rightarrow\ r^n c_n(\max[0,\Delta N]\leqslant n\leqslant N') \tag{27}$$

up to the physically irrelevant overall phase factor. In particular, by adjusting $\lambda=\lambda'$ $(r=1)$, the target state is faithfully reproduced in the range of photon number 0 to N' for the case of $\Delta N\leqslant 0$ $(N'\leqslant N)$. In other words, if the target state is prepared in the range 0 to certain maximal number $\bar{N}$, the probability for its successful teleportation with $\bar{N}\leqslant N'\leqslant N$ is given by

$$Q(\bar{N},\lambda)=\sum_{N'=\bar{N}}^{\infty}\sum_{m'=0}^{N'}\sum_{N=N'}^{\infty}\sum_{m=0}^{N}\frac{P(N,m,\lambda)}{(N+1)(N'+1)}=\lambda^{2\bar{N}}. \tag{28}$$

We finally discuss possible experimental realizations of our method to prepare the MESS's and the number state teleportation with them ($\lambda=\lambda'$ for definiteness). The case of $N=N'=1$ may be regarded as the basis-reduced version of the scheme of Bennett *et al.* [1]. In fact, we have the two Bell states $|N'=1,m'=0\rangle_{1\text{T}}$ and $|N'=1,m'=1\rangle_{1\text{T}}$ rather than four. The number of Bell states is reduced by half if only the EPR resource with $N=1$ is used upon the Bell measurement by Clare. The number-phase measurement can be realized for $N=1$ with beam splitter and phase shift generating a unitary transformation of the modes 2 and 3 as

$$\begin{pmatrix}a_2^\dagger\\ a_3^\dagger\end{pmatrix}=\begin{pmatrix}c & -s\eta\\ s\xi & c\eta\xi\end{pmatrix}_{(23)}\begin{pmatrix}a_{2'}^\dagger\\ a_{3'}^\dagger\end{pmatrix}. \tag{29}$$

By taking $c=1/\sqrt{2}$, $s=1/\sqrt{2}$, $\eta=-1$, $\xi=1$, the Bell states $(N=1,m=0,1)$ are transformed as

$$|1,0\rangle_{23}=|1\rangle_{2'}|0\rangle_{3'}, \quad |1,1\rangle_{23}=|0\rangle_{2'}|1\rangle_{3'}. \tag{30}$$

Then by making the single photon countings on the modes $2'$ and $3'$, we can identify either of the Bell states with $N=1$, obtaining the EPR resource $|1,0\rangle_{14}$ or $|1,1\rangle_{14}$. The same technique can be used on the modes T and 1, and teleportation of a qubit $|\psi\rangle_{\mathrm{T}}=c_0|0\rangle_{\mathrm{T}}+c_1|1\rangle_{\mathrm{T}}$ is completed in the probability $P(1,m,\lambda)=(1-\lambda^2)^2\lambda^2$ (e.g., 0.14 for $\lambda=0.5$), as seen from Eq. (28) with fixed $N=N'=1$. The other results of the photon counters are discarded for this minimal protocol. It is here interesting to mention that based on a similar idea, an experimental result has been reported recently for the teleportation of the vacuum one-photon qubit [9].

The Bell measurements may be done partially even for $N=2$. The point is to introduce an *ancilla* mode $\tilde{a}$. Then by making a series of unitary transformations, such as in Eq. (29) among the modes 2, 3, and $\tilde{a}$, it is possible to realize that only the Bell state $|N=2,m=0\rangle_{23}|0\rangle_{\tilde{a}}$ has the component of $|0\rangle_{2'}|1\rangle_{3'}|1\rangle_{\tilde{a}'}$, while the other Bell states $|N=2,m=1,2\rangle_{23}|0\rangle_{\tilde{a}}$ do not. This means that if we observe the state $|0\rangle_{2'}|1\rangle_{3'}|1\rangle_{\tilde{a}'}$ with single photon detectors, we can identify at least $|N=2,m=0\rangle_{23}|0\rangle_{\tilde{a}}$, producing the EPR resource $|N=2,m=0\rangle_{14}$. The relevant parameters are taken for the sequential transformations as $c=1/\sqrt{2},s=1/\sqrt{2},\eta=1,\xi=1$ for $(2\tilde{a})\rightarrow c=\sqrt{2/3},s=1/\sqrt{3},\eta=1,\xi=(1+i)/\sqrt{2}$ for $(3\tilde{a})\rightarrow c=\sqrt{3/8},s=-\sqrt{5/8},\eta=1,\xi=(3+i)/\sqrt{10}$ for (23). This procedure may be extended for $N>2$.

We believe that the present analysis promotes the theoretical and experimental efforts for the number sum and phase difference measurements, which are essential for realizing the $(N+1)$-dimensional entanglement and teleportation utilizing photons. It is here notable that some ideas and attempts have appeared recently for the phase difference measurement [10]. While the partial Bell measurements may be done with linear operations such as beam splitters and phase shifts, some nonlinear operations will be needed for the perfect Bell measurement, which is one of the very challenging issues in the future for fundamental quantum physics.

In summary, we have formulated the Bell measurement in the photon number basis by describing the simultaneous eigenstates of number sum and phase difference. Then, we have proposed a different method to refine entanglement via swapping from a pair of squeezed vacuum states by performing this number-phase Bell measurement. By adjusting the two squeezing parameters to the same value, these states are maximally entangled. These MESS's can really be used for the reliable teleportation of number states based on the number-phase Bell measurement. We have also discussed some feasible experimental realizations of the number-phase measurement for preparing MESS's and performing teleportation utilizing them, where beam splitters with phase shifts and single-photon counters may be used.

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